Notes in Applied Mathematics

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Chapter 1

Variational mechanics

Our journey will begin looking into the constraints imposed on the system, this should come as no surprise since is by the presence of such constraints that the state of the system can be described by a reduced number of quantities rather than keeping track of all points in the system.

Definition 1.1. *Given a continuum body composed of the material points $\{P_i\}_{i=1}^N$, we define a holonomic constraint as a constraint imposed on the material points that can be expressed as the level set of smooth function, i.e.*

$$f(\boldsymbol{\chi}, t) = 0, \tag{1.1}$$

where $f : \mathbb{R}^{3N} \times [0, \infty) \to \mathbb{R}$ is a smooth function and $\boldsymbol{\chi} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) \coloneqq \bigotimes_{i=1}^N \boldsymbol{x}_i \subset R^{3N}$ is the vector collecting the position of all the material points with respect to fixed reference frame $(O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. Furthermore, we say that a holonomic constraint is a fixed holonomic constraint or scleronomic constraint if it is independent of time, i.e. $f(\boldsymbol{\chi}, t) = f(\boldsymbol{\chi})$.

One of the interesting properties of fixed holonomic constraints is that they identify a submanifold of $\mathbb{R}^{3N}$ of admissible configurations for the system, i.e.

$$\mathcal{M}_{\boldsymbol{f}} = \left\{ \boldsymbol{\chi} \in \mathbb{R}^{3N} \text{ such that } \boldsymbol{f}_j(\boldsymbol{\chi}) = 0 \; \forall 1 \leq j \leq M \right\}, \tag{1.2}$$

where $\boldsymbol{f} : \mathbb{R}^{3N} \to \mathbb{R}$ are the holonomic constraints imposed on the system. It's a known fact if differential geometry that we $\mathcal{M}_{\boldsymbol{f}}$ is a regular surface hence we can parametrise it locally with $\kappa \coloneqq 3N - M$ parameters, i.e.

$$\boldsymbol{\chi} = \boldsymbol{\chi}(q_1, \dots q_\kappa) = \boldsymbol{\chi}\left(\bigotimes_{i=1}^{\kappa} q_i \right), \qquad \forall \boldsymbol{\chi} \in \mathcal{M}_{\boldsymbol{f}}. \tag{1.3}$$

We will call $q_1, \dots q_\kappa$ the *generalized coordinates* of the system, and those are precisely the minimal set of quantities we were looking for in order to describe the state of the system. For brevity, we will denote $\boldsymbol{Q}$ the vector collecting the generalized coordinates of the system, i.e.

$$\boldsymbol{Q} = \bigotimes_{i=1}^{\kappa} q_i. \tag{1.4}$$

1.1 Principal of virtual work

We now begin studying the equilibrium configuration of the system, i.e. the configuration of the system for which the time derivative of the linear momentum vanishes. We observe that if we start from Newton's second law of motion we end up with the following identity

$$\bigotimes_{i=1}^{N} \boldsymbol{F}_i = \bigotimes_{i=1}^{N} m_i \ddot{\boldsymbol{x}}_i = 0, \tag{1.5}$$

where $\boldsymbol{F}_i$ are the forces experience by the material point P_i. The previous equation expresses the Newtonian principle that for a body to be at rest the sum of all forces acting on the body must vanish. We can now decouple the forces acting on the system in two sets, the *active forces* $\boldsymbol{F}_i^a$ and the *constraint forces* $\boldsymbol{\Phi}_i$. These are respectively the external forces acting on the system and the forces acting on the system in order to exert the holonomic constraints,

$$\boldsymbol{F}^{(a)} + \boldsymbol{\Phi} \coloneqq \bigotimes_{i=1}^{N} \boldsymbol{F}_i^a + \bigotimes_{i=1}^{N} \boldsymbol{\Phi}_i = 0. \tag{1.6}$$

The previous equation appears to be a very complicated rephrasing of the rather obvious principle that for a body to be at equilibrium the external forces must be counterbalanced by the constraint forces. Yet we will see that this equation will be the starting point to derive a variational formulation of the dynamics of many physical systems under the following axiom

Axiom 1.1. *We will assume that the constraint forces $\boldsymbol{\Phi}$, corresponding to a configuration $\hat{\boldsymbol{\chi}}$ are orthogonal to the tangent plane to the manifold $\mathcal{M}_{\boldsymbol{f}}$ at $\hat{\boldsymbol{\chi}}$, i.e.*

$$\boldsymbol{\Phi} = \sum_{j=1}^{M} \lambda_j \nabla f_j(\hat{\boldsymbol{\chi}}) = \nabla_{\boldsymbol{\chi}} \boldsymbol{f}(\hat{\boldsymbol{\chi}})^T \boldsymbol{\lambda}, \qquad \forall \hat{\boldsymbol{\chi}} \in \mathcal{M}_{\boldsymbol{f}}. \tag{1.7}$$

We can now rewrite (1.6) making use of the previous axiom, i.e.

$$\boldsymbol{F}^{(a)} + \nabla_{\boldsymbol{\chi}} \boldsymbol{f}(\hat{\boldsymbol{\chi}})^T \boldsymbol{\lambda} = 0, \tag{1.8}$$

this apparently silly equation has great importance, in fact, we can decompose it into two equations on the tangent space to the manifold $\mathcal{M}_{\boldsymbol{f}}$ at $\hat{\boldsymbol{\chi}}$ and one on its orthogonal complement, i.e.

$$\boldsymbol{F}_{\perp}^{(a)} = -\nabla_{\boldsymbol{\chi}} \boldsymbol{f}(\hat{\boldsymbol{\chi}})^T \boldsymbol{\lambda}, \qquad \boldsymbol{F}_{\parallel}^{(a)} = 0, \tag{1.9}$$

which can be expressed using the fact that $\left\{\partial_{q_k} \hat{\boldsymbol{\chi}}\right\}_{k=1}^{\kappa}$ is a basis of the tangent space to $\mathcal{M}_{\boldsymbol{f}}$ at $\hat{\boldsymbol{\chi}}$, i.e.

$$\boldsymbol{F}_{\perp}^{(a)} = -\nabla_{\boldsymbol{\chi}} \boldsymbol{f}(\hat{\boldsymbol{\chi}})^T \boldsymbol{\lambda}, \qquad \sum_{s=1}^{\kappa} \partial_{q_s} \hat{\boldsymbol{\chi}} \left(\boldsymbol{F}^{(a)} \cdot \partial_{q_s} \hat{\boldsymbol{\chi}} \right) = 0. \tag{1.10}$$

The first equation in (1.10) tells us that the constraint forces are uniquely determined by the external forces and that it always exists a set of constraint forces that can counterbalance the external forces, i.e.

$$\boldsymbol{\lambda} = -\nabla_{\boldsymbol{\chi}} \boldsymbol{f}(\hat{\boldsymbol{\chi}})^{-T} \boldsymbol{F}_{\perp}^{(a)}. \tag{1.11}$$

The second equation tells us that the system we are considering is at equilibrium, if and only if for any $1 \leq s \leq \kappa$ the following identity holds

$$\boldsymbol{F}_1^{(a)} \cdot \frac{\partial}{\partial q_s} \hat{\boldsymbol{x}}_1 + \cdots + \boldsymbol{F}_\kappa^{(a)} \cdot \frac{\partial}{\partial q_s} \hat{\boldsymbol{x}}_N = 0 \quad \Rightarrow \quad \left(\nabla_{\boldsymbol{Q}} \hat{\boldsymbol{\chi}}\right)^T \boldsymbol{F}^{(a)} = \boldsymbol{0}. \tag{1.12}$$

This last identity is known as the *principle of virtual work*. It might not be immediate but we managed to derive from the Newtonian principle that states that a body is at rest if and only if the sum, taken with respect to all the points in the system, of all active forces and constraint forces vanishes to a principle which only considers the generalized coordinates of the system. A particularly interesting case is when the right-hand side of (1.12) is an exact differential, i.e.

Definition 1.2. *Given a continuum body composed of the material points $\{P_i\}_{i=1}^N$, we define the set of active forces acting on the continuum body conservative forces or monogenic forces if they can be expressed as the gradient of a scalar function, i.e*

$$\boldsymbol{F} = -\nabla_{\boldsymbol{\chi}} \mathcal{V}(\boldsymbol{\chi}, t) = -\nabla_{\boldsymbol{\chi}} \mathcal{V}\left(\boldsymbol{\chi}(\boldsymbol{Q}), t\right), \quad \forall \hat{\boldsymbol{\chi}} \in \mathbb{R}^{3N} \tag{1.13}$$

where $\mathcal{V} : \mathbb{R}^{3N} \times [0, \infty) \to \mathbb{R}$ is a smooth function known as the potential energy of the system.

Under the assumption that the active forces acting on the system under consideration are monogenic, we can express (1.12) as

$$(\nabla_{\boldsymbol{Q}} \hat{\boldsymbol{\chi}})^T \left(-\nabla_{\boldsymbol{\chi}} \mathcal{V}(\hat{\boldsymbol{\chi}}(\boldsymbol{Q}), t)\right) = -\nabla_{\boldsymbol{Q}} \mathcal{V}(\hat{\boldsymbol{\chi}}(\boldsymbol{Q}), t) = \boldsymbol{0}, \tag{1.14}$$

for every function $\hat{\boldsymbol{\chi}} : \mathbb{R}^\kappa \to \mathcal{M}_{\boldsymbol{f}}$. Therefore we can rephrase the virtual work principle as: "*A continuum body is at equilibrium if and only if the first variation of the potential energy vanishes*". If we assume that the potential energy is a smooth convex function, with respect to $\boldsymbol{Q}$, then (1.14) is equivalent to say that a body is an equilibrium if and only if its potential energy is minimized.

Remark 1.1. *It might not be obvious at first why (1.12) is called a variational principle. The reason is because it must hold as we vary arbitrarily $\hat{\boldsymbol{\chi}}$ within the set of admissible configuration, i.e. it must hold for any $\hat{\boldsymbol{\chi}} : \mathbb{R}^\kappa \to \mathcal{M}_f$.*

Remark 1.2. *It is a well-established fact that a smooth function $\mathcal{V} : \mathbb{R}^{3N} \to \mathbb{R}$ has a stationary point at $\hat{\boldsymbol{\chi}}_0$ if $\nabla_{\boldsymbol{\chi}} \mathcal{V}$ vanishes at $\hat{\boldsymbol{\chi}}_0$. Yet in this case we are considering $\nabla_{\boldsymbol{Q}} \mathcal{V}$ rather than $\nabla_{\boldsymbol{\chi}} \mathcal{V}$, what is the reason behind this choice? Well, the reason is very intuitive. The requirement $\nabla_{\boldsymbol{X}} \mathcal{V}(\hat{\boldsymbol{\chi}}_0, t)$ corresponds to the requirement that if I'm a stationary point and I look to all the points infinitely close to $\hat{\boldsymbol{\chi}}_0$, I should not be able to find a point where $\mathcal{V}$ with a value greater or smaller than $\mathcal{V}(\hat{\boldsymbol{\chi}}_0, t)$, yet in this context, I'm only interested in the point surrounding $\hat{\boldsymbol{\chi}}_0$ that are compatible with the constraints, i.e. $\hat{\boldsymbol{\chi}} \in \mathcal{M}_{\boldsymbol{f}}$. In particular, such points are precisely the ones described by the generalized coordinates, hence we ask $\nabla_{\boldsymbol{Q}} \mathcal{V}(\hat{\boldsymbol{\chi}}_0, t)$ to vanish in order for $\hat{\boldsymbol{\chi}}_0$ and the corresponding $\hat{\boldsymbol{Q}}_0$ to be a stationary point.*

Remark 1.3. *So far we have assumed that we are only interested in stationary points that live in the interior of our configuration space. This has been well motivated by the fact that we assumed our configuration space is the entire $\mathbb{R}^3$, hence we have no boundary, but what happens if this is not the case? Well, things can get much more complicated, first of all, we would have to choose a definition for the notion of the gradient of a function evaluated at a point on the boundary, which is a non-obvious problem. Let's say for example that we assume enough smoothness that we can define the value of the gradient inside of the domain and then perform a regular extension to the boundary, the condition*

$$\nabla_{\boldsymbol{Q}} \mathcal{V}(\hat{\boldsymbol{\chi}}_0, t) = 0, \tag{1.15}$$

is not enough to guarantee that $\hat{\boldsymbol{\chi}}_0$ is a stationary point, since we lack the reversibility property of the gradient when we are on the boundary. We will come back to this point when we will discuss contact mechanics and in geenral problems involving variational inequalities.

Remark 1.4. *We would like to show that the same reasoning that brought us to the virtual work principle expressed as searching for stationary points of the potential energy within the class of admissible configurations, as pointed out in Remark 1.2, can be deduced also from the observation that $\boldsymbol{\lambda}$ acts as the Lagrange multiplier for the holonomic constraints. To achieve this result we focus once again on the first equation of (1.8), and write this identity under the assumption that we are working with monogenic forces, i.e.*

$$\nabla_{\boldsymbol{\chi}} \left(\mathcal{V}(\boldsymbol{\chi}) - \boldsymbol{f}(\boldsymbol{\chi})^T \boldsymbol{\lambda} \right) = \nabla_{\boldsymbol{\chi}} \mathcal{V}(\boldsymbol{\chi}) + \nabla_{\boldsymbol{\chi}} \boldsymbol{f}(\boldsymbol{\chi})^T \boldsymbol{\lambda} = 0 \tag{1.16}$$

Once we realize that the first member of the previous equation is the gradient of the Lagrange function $\overline{\mathcal{V}} \coloneqq \mathcal{V} + \boldsymbol{f}^T \boldsymbol{\lambda}$, hence solving (1.8) is equivalent to looking for stationary points of $\mathcal{V}(\boldsymbol{\chi})$ under the constraints $\boldsymbol{f}(\boldsymbol{\chi}) = 0$, i.e. within the manifold of admissible solutions $\mathcal{M}_{\boldsymbol{f}}$.

1.2 D'Alembert principle

We have so far focused our attention on the equilibrium configuration of the system, hence we focused our attention on the case when the linear moment vanishes, i.e.

$$\boldsymbol{\mathcal{P}} \coloneqq \bigotimes_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i = 0. \tag{1.17}$$

We now wonder what happens when this is not the case. In particular, we consider as a starting point once again Newton's second law applied to all the points constituting a continuum body. In this setting Newton's second law can be stated as the fact that the sum of the forces acting on the system must be in equilibrium. This means that the active forces, constraint forces and inertial forces, i.e. $\bigotimes_{i=1}^{N} m_i \ddot{\boldsymbol{x}}_i$, must be at equilibrium, i.e.

$$\boldsymbol{F}^{(a)} + \boldsymbol{\Phi} = \bigotimes_{i=1}^{N} \boldsymbol{F}_i^{(a)} + \bigotimes_{i=1}^{N} \boldsymbol{\Phi}_i = \dot{\boldsymbol{\mathcal{P}}}, \tag{1.18}$$

rearranging the previous equation we can observe that Newton's second law suggests that the following identity must hold

$$\boldsymbol{F}^{(a)} + \boldsymbol{\Phi} - \dot{\boldsymbol{\mathcal{P}}} = 0. \tag{1.19}$$

This simple rearrangement of the terms in (1.18), suggests that we can apply the virtual work principle considering as forces acting on the continuum body the sum of the active forces and the inertial forces, i.e.

$$(\nabla_{\boldsymbol{Q}} \hat{\boldsymbol{\chi}})^T \left(\boldsymbol{F}^{(a)} - \dot{\boldsymbol{\mathcal{P}}} \right) = \boldsymbol{0}. \tag{1.20}$$

The previous equation is a mathematical formulation of the *d'Alembert principle*, which states that the sum of the active forces and the inertial forces must vanish. At this point, we might be tempted to try express the d'Alembert principle in terms of the potential energy, as we did for the virtual work principle, in order to recast (1.19) as a stationary point problem and express $\boldsymbol{\Phi}$ in terms of the Lagrange multiplier $\boldsymbol{\lambda}$, yet this is not possible. In fact, while we might assume that the active forces are monogenic, we can not assume the same for the inertial forces. There is one noticeable exception to the usual non-monogenic nature of the inertial forces, i.e. when we are considering fixed holonomic constraints. Let us consider ad starting point the kinetic energy of the discrete system of particles, expressed as a function of the velocity corresponding to the generalized coordinates, i.e.

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \left\| \hat{\dot{\boldsymbol{x}}}_i \right\|^2 = \frac{1}{2} \sum_{i=1}^{N} m_i \left\| \dot{\boldsymbol{x}}_i(\hat{\boldsymbol{q}}_i) \right\|^2 = \frac{1}{2} \sum_{i=1}^{N} m_i \left\| \frac{\partial}{\partial t} \boldsymbol{x}_i(\hat{\boldsymbol{q}}_i) + (\nabla_{\boldsymbol{q}_i} \hat{\boldsymbol{x}}_i)^T \hat{\dot{\boldsymbol{q}}}_i \right\|^2, \tag{1.21}$$

where we have denoted $\boldsymbol{q}_i$ the vector containing the generalized coordinates of the i-th particle of the system. When we are considering fixed holonomic constraints, we know that $\frac{\partial}{\partial t} \boldsymbol{x}_i(\hat{\boldsymbol{q}}_i)$ vanishes, hence we can rewrite (1.21) as

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \left\| (\nabla_{\boldsymbol{q}_i} \hat{\boldsymbol{x}}_i)^T \hat{\dot{\boldsymbol{q}}}_i \right\|^2. \tag{1.22}$$

We are now interested in expressing some relevant quantities used in the d'Alembert principle in terms of the kinetic energy, i.e.

$$(\nabla_{\boldsymbol{Q}} \hat{\boldsymbol{\chi}})^T \boldsymbol{\mathcal{P}} = \nabla_{\dot{\boldsymbol{Q}}} T \Big|_{\dot{\boldsymbol{Q}} = \hat{\dot{\boldsymbol{Q}}}}, \qquad (\nabla_{\boldsymbol{Q}} \hat{\dot{\boldsymbol{\chi}}})^T \boldsymbol{\mathcal{P}} = \nabla_{\boldsymbol{Q}} T \Big|_{\boldsymbol{Q} = \hat{\boldsymbol{Q}}}. \tag{1.23}$$

We can compute the material time derivative of the first identity in (1.23) to obtain,

$$(\nabla_{\boldsymbol{Q}} \hat{\dot{\boldsymbol{\chi}}})^T \boldsymbol{\mathcal{P}} + (\nabla_{\boldsymbol{Q}} \hat{\boldsymbol{\chi}})^T \dot{\boldsymbol{\mathcal{P}}} = D_t \left(\nabla_{\dot{\boldsymbol{Q}}} T \right), \tag{1.24}$$

we now substitute the previous identity in (1.20) to obtain a version of the d'Alambert principle expressed in terms of the kinetic energy, i.e.

$$(\nabla_{\boldsymbol{Q}} \hat{\boldsymbol{\chi}})^T \boldsymbol{F}^{(a)} - D_t \left(\nabla_{\dot{\boldsymbol{Q}}} T \right) + (\nabla_{\boldsymbol{Q}} \hat{\dot{\boldsymbol{\chi}}})^T \boldsymbol{\mathcal{P}} = (\nabla_{\boldsymbol{Q}} \hat{\boldsymbol{\chi}})^T \boldsymbol{F}^{(a)} - D_t \left(\nabla_{\dot{\boldsymbol{Q}}} T \right) + \nabla_{\boldsymbol{Q}} T = 0, \tag{1.25}$$

The last equation can also be expressed component-wise and we will refer to this set of equations as *Euler–Lagrange equations*, i.e.

$$D_t \frac{\partial}{\partial \dot{q}_s} T - \frac{\partial}{\partial q_s} T = \sum_{i=1}^{\kappa} \boldsymbol{F}_i^{(a)} \cdot \frac{\partial}{\partial q_s} \hat{\boldsymbol{x}}_i, \qquad \forall 1 \leq s \leq \kappa. \tag{1.26}$$

So far we haven't made any assumption on the nature of the active forces, but if we assume that they are conservative, i.e. $\boldsymbol{F}^{(a)} = -\nabla_{\boldsymbol{\chi}} \mathcal{V}(\boldsymbol{\chi})$, we can rewrite (1.25) as

$$\begin{aligned} &D_t \left(\nabla_{\dot{\boldsymbol{Q}}} T\right) - \nabla_{\boldsymbol{Q}} T - (\nabla_{\boldsymbol{Q}} \hat{\boldsymbol{\chi}})^T \nabla_{\boldsymbol{\chi}} \mathcal{V}(\hat{\boldsymbol{\chi}}) = 0, \\ &D_t \left(\nabla_{\dot{\boldsymbol{Q}}} T\right) - \nabla_{\boldsymbol{Q}} \left(T + \mathcal{V}\right) = 0. \end{aligned} \tag{1.27}$$

We can introduce the *Lagrangian function* of the system as the difference between the kinetic energy and the potential energy and rewrite the previous equation as

$$D_t \left(\nabla_{\dot{\boldsymbol{Q}}} \mathcal{L}\right) - \nabla_{\boldsymbol{Q}} \mathcal{L} = 0, \quad \mathcal{L} \coloneqq T - \mathcal{V}. \tag{1.28}$$

Remark 1.5. *When deriving the Euler–Lagrange equations we have assumed that $\mathcal{L}$ is a function of both the generalized coordinates and the generalized velocities, i.e. $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, yet we know that $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are somehow related. The reasoning that motivates us to consider $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ as independent variables is that when we are introducing the generalized coordinates we essentially eliminating the dependence on time and expressing the quantities we are interested in terms of the generalized coordinates. This is done because the space of admissible configuration can be parametrised by the generalized coordinates, the same can be said for the space of admissible velocities, hence we can consider $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ as independent variables. By differentiating the constraint with respect to time and taking advantage of the scleronomic nature of the constraint, we obtain*

$$\nabla_{\boldsymbol{\chi}} \boldsymbol{f}(\hat{\boldsymbol{\chi}})^T \dot{\boldsymbol{\chi}} = 0, \tag{1.29}$$

suggesting that $\dot{\boldsymbol{\chi}}$ lives in the tangent space to the manifold of admissible configurations. In particular, we can easily observe that,

$$\dot{\boldsymbol{\chi}} = \frac{\partial}{\partial t} \boldsymbol{\chi} + \sum_{i=1}^{\kappa} \frac{\partial}{\partial q_i} \boldsymbol{\chi}\, \dot{q}_i = \frac{\partial}{\partial t} \boldsymbol{\chi} + \nabla_{\boldsymbol{Q}} \boldsymbol{\chi}^T \dot{\boldsymbol{Q}}, \tag{1.30}$$

hence we can express all the admissible values of $\dot{\boldsymbol{\chi}}$ varying $\dot{\boldsymbol{Q}}$ arbitrarily in $\mathbb{R}^{\kappa}$ and $\frac{\partial}{\partial t}\boldsymbol{\chi}$ has to vanish.

Remark 1.6. *An important fact to notice is that we have explicitly made use of the fact that the active forces are conservative and only depend on the generalized coordinates and not on the corresponding velocities. We could also have taken into consideration the case when the active forces depend also on the generalized velocities and still obtain (1.28). An in-depth discussion can be found in [1, Chapter 4.7].*

Remark 1.7. *The Euler–Lagrange equation can be rewritten to highlight its connection with Newton's second law, i.e.*

$$D_t \left(\nabla_{\dot{\boldsymbol{Q}}} T\right) - \nabla_{\boldsymbol{Q}} \mathcal{T} = D_t(\nabla_{\dot{\boldsymbol{Q}}} \mathcal{V}) - \nabla_{\boldsymbol{Q}} \mathcal{V}, \tag{1.31}$$

where the left-hand side of the previous equation represents the energy connected to the inertial forces and the right-hand side represents the active forces.

It is now time to introduce the reader to perhaps on of the most important results in variational mechanics, i.e. the *Nöther principle* which can be roughly stated as: "*If a Lagrangian is invariant under a transformation, then there is a corresponding quantity that is conserved during the motion described by the Euler–Lagrange equations*". Of course when formalizing a principle as the one previously stated we need to be more precise on what we mean by invariant under a transformation.

Definition 1.3. *We define a symmetry of a Lagrangian $\mathcal{L}$ a transformation*

$$\boldsymbol{S} : \mathbb{R}^\kappa \times \mathbb{R} \to \mathbb{R}^\kappa \qquad \hat{\boldsymbol{Q}} \mapsto \boldsymbol{S}(\hat{\boldsymbol{Q}}, s), \tag{1.32}$$

if the following requirements are satisfied:

1. *The Lagrangian $\mathcal{L}$ is invariant under the transformation S, i.e.*

$$\mathcal{L}(\boldsymbol{S}(\hat{\boldsymbol{Q}}, s), \dot{\boldsymbol{S}}(\hat{\boldsymbol{Q}}, s)) = \mathcal{L}(\hat{\boldsymbol{Q}}, \dot{\hat{\boldsymbol{Q}}}), \qquad \forall \hat{\boldsymbol{Q}} \in \mathbb{R}^\kappa,\ \forall s \in \mathbb{R}. \tag{1.33}$$

2. *Zero is a fixed parameter of the transformation S, i.e.*

$$\boldsymbol{S}(\hat{\boldsymbol{Q}}, 0) = \hat{\boldsymbol{Q}}, \qquad \forall \hat{\boldsymbol{Q}} \in \mathbb{R}^\kappa. \tag{1.34}$$

3. *The transformation S is compossible in the following manner,*

$$\boldsymbol{S}(\boldsymbol{S}(\hat{\boldsymbol{Q}}, s), s') = \boldsymbol{S}(\hat{\boldsymbol{Q}}, s + s'), \qquad \forall \hat{\boldsymbol{Q}} \in \mathbb{R}^\kappa,\ \forall s, s' \in \mathbb{R}. \tag{1.35}$$

Using the previous definition of symmetry we can formalize the Nöther principle in the following theorem.

Theorem 1.1. *Let $\mathcal{L}$ be a Lagrangian and $\boldsymbol{S} : \mathbb{R}^\kappa \times \mathbb{R} \to \mathbb{R}^\kappa$ a symmetry of $\mathcal{L}$, then the quantity*

$$\mathcal{I}(\hat{\boldsymbol{Q}}, \hat{\dot{\boldsymbol{Q}}}) \coloneqq \nabla_{\dot{\boldsymbol{Q}}} \mathcal{L} \cdot \frac{\partial}{\partial s} \boldsymbol{S} \Big|_{s=0} \tag{1.36}$$

is conserved along any admissible motion of the system.

Proof. We begin computing the derivative of $\mathcal{L}\left(\boldsymbol{S}(\boldsymbol{Q}, s)\right), \dot{\boldsymbol{S}}(\boldsymbol{Q}, s)$ with respect to s and observing that since $\boldsymbol{S}$ is a symmetry such derivative has to vanish, i.e.

$$\frac{\partial}{\partial s} \mathcal{L}\left(\boldsymbol{S}, \dot{\boldsymbol{S}}\right) = \nabla_{\boldsymbol{Q}} \mathcal{L}(\boldsymbol{S}, \dot{\boldsymbol{S}}) \frac{\partial}{\partial s} \boldsymbol{S} + \nabla_{\dot{\boldsymbol{Q}}} \mathcal{L}(\boldsymbol{S}, \dot{\boldsymbol{S}}) \frac{\partial}{\partial s} \dot{\boldsymbol{S}} = 0. \tag{1.37}$$

We now consider the Euler–Lagrange equations (1.28) and multiply both sides by $\partial_s \boldsymbol{S}$ to obtain

$$D_t \left(\nabla_{\dot{\boldsymbol{Q}}} \mathcal{L}\right) \cdot \frac{\partial}{\partial s} \boldsymbol{S} - \nabla_{\boldsymbol{Q}} \mathcal{L} \cdot \frac{\partial}{\partial s} \boldsymbol{S} = 0, \tag{1.38}$$

hence we can expand the total derivative of $\nabla_{\dot{\boldsymbol{Q}}} \mathcal{L} \cdot \frac{\partial}{\partial s} \boldsymbol{S}$ to obtain

$$\begin{aligned} D_t \left(\nabla_{\dot{\boldsymbol{Q}}} \mathcal{L} \cdot \frac{\partial}{\partial s} \boldsymbol{S} \right) &= \frac{\partial}{\partial s} \boldsymbol{S} \cdot D_t \left(\nabla_{\dot{\boldsymbol{Q}}} \mathcal{L}\right) + \nabla_{\dot{\boldsymbol{Q}}} \mathcal{L} \cdot D_t \left(\frac{\partial}{\partial s} \boldsymbol{S} \right) \\ &= \frac{\partial}{\partial s} \boldsymbol{S} \cdot \nabla_{\boldsymbol{Q}} \mathcal{L} + \nabla_{\dot{\boldsymbol{Q}}} \mathcal{L} \cdot \frac{\partial}{\partial s} \dot{\boldsymbol{S}} = 0. \end{aligned} \tag{1.39}$$

□

Corollary 1.1. *Let $\mathcal{L}$ be a Lagrangian and $\boldsymbol{S} : \mathbb{R}^\kappa \times \mathbb{R} \to \mathbb{R}^\kappa$ a symmetry of $\mathcal{L}$. Furthermore let G be infinitesimal generator of the transformation $\boldsymbol{S}$, i.e. $G(\hat{\boldsymbol{Q}}) = \frac{\partial}{\partial s}\boldsymbol{S}(\hat{\boldsymbol{Q}}, s)\Big|_{s=0}$, then*

$$\frac{d}{dt}\left(\nabla_{\dot{\boldsymbol{Q}}}\mathcal{L} \cdot G(\hat{\boldsymbol{Q}})\right) = 0, \tag{1.40}$$

Remark 1.8. *We can now derive the conservation of linear and angular momentum, for a continuum rigid body, from the Nöther principle, in fact it is sufficient to notice that the Lagrangian of a rigid body has the following symmetries,*

$$\boldsymbol{S}_1(\boldsymbol{q}, \boldsymbol{\alpha}, s) = (\boldsymbol{q} + s\boldsymbol{\delta q}, \boldsymbol{\alpha}), \qquad \boldsymbol{S}_2(\boldsymbol{q}, \boldsymbol{\alpha}, s) = (\boldsymbol{q}, \boldsymbol{\alpha} + s\boldsymbol{\delta\alpha}). \tag{1.41}$$

1.3 Canonical Formalism

The Euler–Lagrange equations that we have derived in the previous section are a system of second-order differential equations, in fact, the term $D_t\left(\nabla_{\dot{\boldsymbol{Q}}}\mathcal{L}\right)$ yields in a second order derivative with respect to time. A fairly standard technique to solve a system of κ second-order differential equations is to reduce the order of the system by introducing κ new variables and obtain a system of 2κ first-order differential equations. In the particular case of (1.28) we need to introduce as new variable the generalized momenta, i.e.

$$\boldsymbol{P} = \bigotimes_{i=1}^{\kappa} p_i, \qquad p_i \coloneqq \nabla_{\dot{q}_i}\mathcal{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}), \quad \forall 1 \leq i \leq \kappa. \tag{1.42}$$

Remark 1.9. *It is important not to confuse $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{P}$. In fact, while both vectors represent in some sense the linear momentum they are not the same object.*

How can we now transform the previous system of second-order differential equations into a system of first-order differential equations involving the generalized coordinates and the generalized momenta? To answer this question the Legendre transform comes to our aid.

Definition 1.4. *Given a proper function $f : \mathbb{R}^\kappa \to \mathbb{R}$ we define the Legendre-Fenchel transform of f as the function $f^* : \mathbb{R}^\kappa \to \mathbb{R}$ defined as*

$$f^*(\boldsymbol{x}^*) \coloneqq \sup_{\boldsymbol{x}\in\mathbb{R}^\kappa} \left\{(\boldsymbol{x}^*)^T\boldsymbol{x} - f(\boldsymbol{x})\right\}. \tag{1.43}$$

One interesting property of the Legendre-Fenchel transform is that if f is convex and smooth, we require $f \in C^2(\mathbb{R}^\kappa)$, then we can write an equation for the value of $\boldsymbol{x}$ satisfying the supremum condition, i.e.

$$\nabla_{\boldsymbol{x}}\left((\boldsymbol{x}^*)^T\boldsymbol{x} - f(\boldsymbol{x})\right) = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{x}^* = \nabla_{\boldsymbol{x}} f(\boldsymbol{x}), \tag{1.44}$$

hence if we choose $\boldsymbol{x}^* = \nabla_x f(\boldsymbol{x})$ we can rewrite the Legendre-Fenchel transform as

$$f^*(\boldsymbol{x}^*) = (\boldsymbol{x}^*)^T\boldsymbol{x} - f(\boldsymbol{x}). \tag{1.45}$$

It remains to observe that since $\boldsymbol{x}^* = \nabla_{\boldsymbol{x}} f(\boldsymbol{x})$ by Dini's theorem we can write $\boldsymbol{x} = g(\boldsymbol{x}^*)$ for a certain function g and thus rewrite the Legendre-Fenchel transform as a function of $\boldsymbol{x}^*$ only.

Remark 1.10. *We can think of the proper convex function $f : \mathbb{R}^\kappa \to \mathbb{R}$ as a function of $\boldsymbol{x}$ and a parameter $\boldsymbol{y}$, i.e. $f(\boldsymbol{x}; \boldsymbol{y})$, and consider the Legendre-Fenchel transform of f with respect to $\boldsymbol{x}$, i.e.*

$$f^*(\boldsymbol{x}^*; \boldsymbol{y}) \coloneqq \sup_{\boldsymbol{x} \in \mathbb{R}^\kappa} \left\{ (\boldsymbol{x}^*)^T \boldsymbol{x} - f(\boldsymbol{x}; \boldsymbol{y}) \right\}. \tag{1.46}$$

where we define $\boldsymbol{x}^* = \nabla_{\boldsymbol{x}} f(\boldsymbol{x}; \boldsymbol{y})$. As before, we can apply Dini's theorem to express the Legendre-Fenchel transform f^* in terms of $\boldsymbol{x}^*$ if f is now convex in both $\boldsymbol{x}$ and $\boldsymbol{y}$ or if f is convex in $\boldsymbol{x}$ and doesn't depend on $\boldsymbol{y}$.

Now we proceed to apply the Legendre transformation to the Lagrangian $\mathcal{L}$, in particular, we consider the Legendre-Fenchel transform of $\dot{\boldsymbol{Q}} \mapsto \mathcal{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$ keeping $\boldsymbol{Q}$ fixed, i.e.

$$\mathcal{L}^*(\boldsymbol{Q}, \boldsymbol{P}) \coloneqq \sup_{\dot{\boldsymbol{Q}} \in \mathbb{R}^\kappa} \left\{ \dot{\boldsymbol{Q}}^T \boldsymbol{P} - \mathcal{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}) \right\}. \tag{1.47}$$

Now notice that since we wanted to reduce the order of the system of differential equations we aimed at expressing the previous equation in terms of $\boldsymbol{Q}$ and $\boldsymbol{P}$, $\boldsymbol{P}$ being $\boldsymbol{P} = \nabla_{\dot{\boldsymbol{Q}}} \mathcal{L}$ so that we can rewrite the previous transformation as

$$\mathcal{L}^*(\boldsymbol{Q}, \boldsymbol{P}) = \boldsymbol{P}^T \dot{\boldsymbol{Q}} - \mathcal{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}). \tag{1.48}$$

Definition 1.5. *Given a convex Lagrangian $\mathcal{L}$ we define the corresponding Hamiltonian as the Legendre-Fenchel transform of $\dot{\boldsymbol{Q}} \mapsto \mathcal{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$, i.e.*

$$\mathcal{H}(\boldsymbol{Q}, \boldsymbol{P}) \coloneqq \sup_{\dot{\boldsymbol{Q}} \in \mathbb{R}^\kappa} \left\{ \dot{\boldsymbol{Q}}^T \boldsymbol{P} - \mathcal{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}) \right\} = \boldsymbol{P}^T \dot{\boldsymbol{Q}} - \mathcal{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}), \tag{1.49}$$

where $\boldsymbol{P}$ is the generalized momentum defined as $\boldsymbol{P} = \bigotimes_{i=1}^{\kappa} p_i$, with $p_i = \nabla_{\dot{q}_i} \mathcal{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$.

We now notice that there is a certain symmetry between the derivatives of Lagrangian and the Hamiltonian, i.e.

$$\nabla_{\boldsymbol{Q}} \mathcal{L} = -\nabla_{\boldsymbol{Q}} \mathcal{H} \qquad \Rightarrow \qquad \dot{\boldsymbol{P}} = -\nabla_{\boldsymbol{Q}} \mathcal{H}, \tag{1.50}$$

where the last equation comes from (1.28) using the first identity in (1.50). Furthermore, we can easily observe that since the Legendre transformation is as involution, the proof of this properties can be found in [1], therefore we can express $\dot{\boldsymbol{Q}}$ as

$$\nabla_{\boldsymbol{P}} \mathcal{H} = \dot{\boldsymbol{Q}}. \tag{1.51}$$

Combining (1.50) and (1.51) we obtain the following system of first-order differential equations,

$$\dot{\boldsymbol{Q}} = \nabla_{\boldsymbol{P}} \mathcal{H}, \qquad \dot{\boldsymbol{P}} = -\nabla_{\boldsymbol{Q}} \mathcal{H}. \tag{1.52}$$

The previous equations are known as *Hamilton's equations* and they are equivalent to the Euler–Lagrange equations (1.28).

Remark 1.11. *Notice that in Remark 1.6 we discussed the possibility that the potential $\mathcal{V}$ depends also on the generalized velocities and concluded that the Euler–Lagrange equations (1.28) still hold. Analogously it can also be proven that the Hamiltonian equations still hold.*

1.4 The phase space

When we introduced the generalized coordinates we did so to simplify the description of the admissible configurations of a mechanical system, in particular, we wanted to parametrise the admissible configurations manifold. Achieving this we have implicitly introduced a new space, the so-called *configuration space*, which is the space of all the admissible configurations of a mechanical system. In **remark 1.5** we have also introduced the notion of generalized velocities and observed that they can be treated as independent variables from the generalized coordinates. We are now interested in describing the manifold of admissible configurations in terms of the generalized coordinates and the generalized velocities, but instead of considering the generalized velocities we will use the closely related concept of generalized momenta. The switch from generalized velocities to generalized momenta is motivated by the fact that the generalized momenta are the natural variables to consider when using Hamiltonian formalism.

Definition 1.6. *We define the phase space of a mechanical system as the space $\mathbb{R}^{2\kappa}$, where κ is the number of degrees of freedom of the system. Furthermore we denote the points in the phase space as $\mathbf{\Gamma} = \bigotimes_{i=1}^{\kappa}(\boldsymbol{q}_i, \boldsymbol{p}_i)$.*

We can now think of (1.52) as describing a flow in the phase space. In the sense that we can fix a particle $(\boldsymbol{Q}_0, \boldsymbol{P}_0)$ in the phase space and track its evolution in time integrating Hamilton's equations. In particular, we can think of the solution of Hamilton's equations as a motion of the phase space, i.e. $\boldsymbol{\varphi} : \mathbb{E}^{2\kappa} \times [0, \infty) \to \mathbb{E}^{2\kappa}$.

Remark 1.12. *We would like to point out that even if the phase-space is subject to motions as a continuum body we should not think of the phase space as a continuum body. The phase space is a mathematical abstraction that we use to describe the admissible configurations of a mechanical system, hence we should not think of it as a physical object. This difference between the phase space and a continuum body should be obvious if we think about the cardinality of the points composing the two objects, in fact, the phase space is composed of a continuous number of points while a continuum body is composed of a countable number of points. To highlight the difference between a continuum body and the phase space we will use the term phase continuum to refer to the idea of a phase space, that evolves in time under the effect of a motion $\boldsymbol{\varphi} : \mathbb{E}^{2\kappa} \times [0, \infty) \to \mathbb{E}^{2\kappa}$, as a continuum body does.*

Now that we have introduced the notion of phase space and phase continuum it's time to characterise the motion $\boldsymbol{\varphi} : \mathbb{E}^{2\kappa} \times [0, \infty) \to \mathbb{E}^{2\kappa}$ that are compatible with Hamilton's equations. We begin observing computing the total material derivative of the Hamiltonian as it undergoes the motion $\boldsymbol{\varphi}$, i.e.

$$D_t \mathcal{H}(\mathbf{\Gamma}) = \frac{\partial}{\partial t}\mathcal{H}(\mathbf{\Gamma}) + \nabla_{\mathbf{\Gamma}}\mathcal{H}(\mathbf{\Gamma}) \cdot \dot{\mathbf{\Gamma}} = \nabla_{\boldsymbol{Q}}\mathcal{H}(\mathbf{\Gamma}) \cdot \dot{\boldsymbol{Q}} + \nabla_{\boldsymbol{P}}\mathcal{H}(\mathbf{\Gamma}) \cdot \dot{\boldsymbol{P}}. \tag{1.53}$$

Notice that the term $\frac{\partial}{\partial t}\mathcal{H}(\Gamma)$ vanishes because as previously discussed the Hamiltonian doesn't depend explicitly on time since we are only considering fixed holonomic constraints. Combining (1.53) together with (1.52) we immediately see that the material time derivative of the Hamiltonian vanishes. Hence any particle starting its motion in the phase space on the level set $\mathcal{H}(\mathbf{\Gamma}) = c$ will remain on the same level set for all times.

While this might seem a futile geometrical observation it is actually a very important result from a physical point of view, in fact, it means that the energy of a mechanical system is conserved along any admissible motion if the system is subject to holonomic constraints and the kinetic energy is a homogeneous positive definite quadratic form of the generalized velocities. In fact, if the kinetic energy is a homogeneous positive definite quadratic form of the generalized velocities we can observe that

$$T = \frac{1}{2}\dot{\boldsymbol{Q}}^T A_T \dot{\boldsymbol{Q}} = \frac{1}{2}\nabla_{\dot{\boldsymbol{Q}}} T \cdot \dot{\boldsymbol{Q}}. \tag{1.54}$$

where A_T is a positive definite matrix. Making use of the previous identity we can rewrite the Hamiltonian as

$$\mathcal{H}(\boldsymbol{\Gamma}) = \boldsymbol{P}^T\dot{\boldsymbol{Q}} - \mathcal{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}) = \nabla_{\dot{\boldsymbol{Q}}} T \cdot \dot{\boldsymbol{Q}} = 2T(\dot{\boldsymbol{Q}}) - T(\dot{\boldsymbol{Q}}) + \mathcal{V}(\boldsymbol{Q}) = T(\dot{\boldsymbol{Q}}) + \mathcal{V}(\boldsymbol{Q}), \tag{1.55}$$

since the sum of the kinetic energy and the potential energy of the system represents the all energy we can conclude that if the kinetic energy is a homogeneous positive definite quadratic form of the generalized velocities then the Hamiltonian is the total energy of the system.

Another extremely interesting property of the phase continuum is that it is incompressible, such a result takes the name of *Liouville's theorem* and it is a direct consequence of (1.52).

Theorem 1.2. *Let $\boldsymbol{\varphi} : \mathbb{E}^{2\kappa} \times [0, \infty) \to \mathbb{E}^{2\kappa}$ be a motion of the phase continuum, with smooth Hamiltonian $\mathcal{H}$, then the velocity field $\boldsymbol{v} : \mathbb{E}^{2\kappa} \times [0, \infty) \to \mathbb{R}^{2\kappa}$ associated with $\boldsymbol{\varphi}$ is incompressible, i.e. $\nabla \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{\Gamma}$.*

Proof. We begin by computing the divergence of the velocity field $\boldsymbol{v}(\boldsymbol{\Gamma})$, i.e.

$$\nabla \cdot \boldsymbol{v} = \operatorname{tr}\left(\nabla_{\boldsymbol{\Gamma}} \dot{\boldsymbol{\Gamma}}\right) = \sum_{i=1}^{\kappa} \frac{\partial}{\partial q_i}\dot{q}_i + \frac{\partial}{\partial p_i}\dot{p}_i = \sum_{i=1}^{\kappa} \frac{\partial}{\partial q_i}\frac{\partial}{\partial p_i}\mathcal{H} - \frac{\partial}{\partial p_i}\frac{\partial}{\partial q_i}\mathcal{H}, \tag{1.56}$$

it remains to observe that since we assumed the Hamiltonian is smooth we can invert the order of differentiation of one of the two terms in the last sum and conclude that $\nabla \cdot \boldsymbol{v} = 0$. □

Let's now go back to the concept of phase continuum and in analogy to what is usally done for a continuum body we introduce a continuous function $\pi : \mathbb{R}^{2\kappa} \to \mathbb{R}$ which represents a distribution of particles in the phase space. We can then apply Liouville's theorem to prove that along the trajectory described by (1.52) the distribution π is constant, i.e.

Corollary 1.2. *Let $\boldsymbol{\varphi} : \mathbb{E}^{2\kappa} \times [0, \infty) \to \mathbb{E}^{2\kappa}$ be a motion of the phase continuum, with smooth Hamiltonian $\mathcal{H}$, and $\pi : \mathbb{R}^{2\kappa} \to \mathbb{R}$ a continuous distribution of particles in the phase space, then*

$$D_t\pi = \frac{\partial}{\partial t}\pi + \nabla_{\boldsymbol{\Gamma}}\pi \cdot \boldsymbol{v} = 0 \tag{1.57}$$

Proof. We begin considering any volume $\hat{\mathcal{S}}$ in the phase space and observing any variation in the mass of the volume $\hat{\mathcal{S}}$ must be either the result of the variation of the density $\pi : \mathbb{R}^{2\kappa} \to \mathbb{R}$ or the result of the flux of particles through the boundary of $\hat{\mathcal{S}}$, i.e.

$$\frac{\partial}{\partial t}\int_{\hat{\mathcal{S}}} \pi(\boldsymbol{\Gamma})\, d\boldsymbol{\Gamma} = \int_{\hat{\mathcal{S}}} \frac{\partial}{\partial t}\pi(\boldsymbol{\Gamma})\, d\boldsymbol{\Gamma} = -\int_{\partial\hat{\mathcal{S}}} \boldsymbol{J}\cdot\boldsymbol{n}\, d\boldsymbol{\Gamma}, \tag{1.58}$$

where $\boldsymbol{n}$ is the normal, pointing outwards, to the manifold $\partial\hat{\mathcal{S}}$ and $\boldsymbol{J}$ is the current associated with (1.52), i.e.

$$\boldsymbol{J} : \hat{\mathcal{S}} \times [0,\infty) \to \mathbb{R}^{2\kappa} \qquad \boldsymbol{\Gamma} \mapsto \pi(\boldsymbol{\Gamma})\boldsymbol{v}(\boldsymbol{\Gamma}). \tag{1.59}$$

Now we observe that we can use the divergence theorem to rewrite the previous equation as

$$\int_{\hat{\mathcal{S}}} \frac{\partial}{\partial t}\pi(\boldsymbol{\Gamma})\, d\boldsymbol{\Gamma} = -\int_{\hat{\mathcal{S}}} \boldsymbol{J}\cdot\boldsymbol{n}\, d\boldsymbol{\Gamma} = \int_{\hat{\mathcal{S}}} \nabla_{\boldsymbol{\Gamma}}\cdot\boldsymbol{J}\, d\boldsymbol{\Gamma}, \tag{1.60}$$

since the previous equation holds for any volume $\hat{\mathcal{S}}$, under the assumption we are working with smooth enough functions, we can conclude that

$$\frac{\partial}{\partial t}\pi(\boldsymbol{\Gamma}) + \nabla_{\boldsymbol{\Gamma}}\cdot\boldsymbol{J} = 0. \tag{1.61}$$

We are now left computing the divergence of the current $\boldsymbol{J}$, i.e.

$$\nabla_{\boldsymbol{\Gamma}}\cdot\boldsymbol{J} = \nabla_{\boldsymbol{\Gamma}}\pi\cdot\boldsymbol{v} + \pi\nabla_{\boldsymbol{\Gamma}}\cdot\boldsymbol{v}, \tag{1.62}$$

where the last term vanishes by virtue of Liouville's theorem. We can then conclude that

$$D_t\pi = \frac{\partial}{\partial t}\pi(\boldsymbol{\Gamma}) + \nabla_{\boldsymbol{\Gamma}}\pi\cdot\boldsymbol{v} = 0. \tag{1.63}$$

□

Bibliography

[1] A. Fasano, S. Marmi, and B. Pelloni. *Analytical Mechanics : An Introduction*. Oxford graduate texts. Oxford University Press, 2013.